

## Nonlinear supersymmetric $\sigma$ model for scalar classical waves

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(Received 20 June 1997)

We derive a nonlinear supersymmetric  $\sigma$  model for the transport of light (classical waves) through a disordered medium. We compare this model with the well-established nonlinear  $\sigma$  model for the transport of electrons (Schrödinger waves) and display similarities of and differences between both cases. We show that for weak disorder both models are equivalent (have the same effective Lagrangian). This effective Lagrangian correctly reproduces the (different) Ward identities for Schrödinger waves and for classical waves.  
[S1063-651X(98)04202-0]

PACS number(s): 05.45.+b, 78.20.Bh

### I. INTRODUCTION AND MOTIVATION

In recent years, the transport of electrons and light through disordered media has been studied intensely and many interesting effects have been observed and understood. Examples are universal conductance fluctuations and weak localization for electrons and speckle patterns and enhanced backscattering for light. A thorough discussion may be found in Refs. [1,2]. The universal tool to deal with these and other phenomena in the case of electrons has been Efetov's supersymmetric nonlinear  $\sigma$  model (SUSIG) [3]. This model successfully describes not only the perturbative effects mentioned above but also nonperturbative features such as localization. It correctly accounts for both transport properties and spectral fluctuations. Thus it is fair to say that SUSIG embodies the essence of electronic properties of disordered media. SUSIG so far has not been extended to the transmission of light through disordered media. In the present paper, we aim at filling this gap. Our motivation for this work is the following.

The transmission of light through disordered media is commonly described in terms of the scalar wave equation rather than a variant of Maxwell's equations [2]. The scalar wave equation differs in a fundamental way from the Schrödinger equation for electrons; see Sec. II. By the same token, the Ward identities for both equations differ substantially. In trying to extend SUSIG to the scalar wave equation, we probe the ability of the nonlinear  $\sigma$  model to provide a universal description of wave propagation in disordered systems described by different wave equations. It is of interest to see in which way the difference in wave equations is reflected in the effective Lagrangian of the resulting SUSIG. We will show that SUSIG does apply to the scalar wave equation and that Efetov's effective Lagrangian is universal: It has the same form for Schrödinger waves and for scalar waves. Using the replica trick, John and Stephen [4] derived a nonlinear  $\sigma$  model for classical waves. This derivation was confined, however, to waves at fixed energy and thus bypassed the crucial issue of correlations between amplitudes at *different* energies. The latter play the central role in SUSIG.

The paper is organized as follows. In Sec. II we compare the wave equations for electrons and classical waves and derive the simplest variant of the Ward identities for both.

Section III describes the derivation of nonlinear  $\sigma$  model for classical waves in the supersymmetric formalism. We compare our result with the analogous expression in Efetov's work [3]. Section IV is devoted to the derivation of the Ward identities within a supersymmetric formalism for classical waves. Our conclusions are presented in Sec. V.

### II. WAVE EQUATIONS

We set the mass  $m$  of the electron and Planck's constant  $\hbar$  equal to unity. The Schrödinger equation for a noninteracting electron

$$\left(-\frac{1}{2}\Delta + V(r)\right)\phi = E\phi \quad (1)$$

contains the random potential  $V(r)$ , which describes impurity scattering. The propagation of light is described by the classical wave equation

$$[-\Delta + k^2\delta\epsilon(\mathbf{r})]\phi = k^2\phi. \quad (2)$$

Here  $k = \omega/c$  is the wave number. We have decomposed the space-dependent dielectric constant  $\epsilon(r) = 1 - \delta\epsilon(\mathbf{r})$  into a space-independent background term (which we set equal to 1) and a fluctuating part  $\delta\epsilon(\mathbf{r})$ . We assume that  $\delta\epsilon$  is a Gaussian random process with vanishing first moment and a second moment given by

$$\langle \delta\epsilon(\mathbf{r}_1)\delta\epsilon(\mathbf{r}_2) \rangle = \frac{4\pi}{\mathcal{L}k^{d+1}}\delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (3)$$

where  $\mathcal{L}$  is the elastic mean free path and  $d$  is the dimension of the system. We have written Eq. (3) in complete analogy to the case of electrons.

We compare Eqs. (1) and (2). Aside from a factor 2, the quantity  $k^2$  corresponds formally to the energy  $E$ . However,  $k^2$  is always positive, in contradistinction to  $E$ . The main difference between Eqs. (1) and (2) lies in the energy dependence of the random potential in the classical case. While  $V$  is independent of energy, the analogous term  $\sim \delta\epsilon$  is not. This difference is also reflected in different Ward identities

that relate averaged one- and two-point functions. For electrons, the retarded (advanced) Green's function

$$G_e^\pm = (E^\pm \pm z + \Delta/2 - V)^{-1} \quad (4)$$

is taken at energies  $E^+ + z = E + z + i\eta$  and  $E^- - z = E - z - i\eta$ , respectively. We immediately find

$$\langle G_e^+ \rangle - \langle G_e^- \rangle = -2(z + i\eta) \langle G_e^+ G_e^- \rangle, \quad (5)$$

where the angular brackets stand for the ensemble average. In complete analogy, we define the Green's functions for classical waves by

$$G_c^\pm = \left[ \left( k_0^2 \pm \frac{\Delta k^2}{2} \right) + \Delta - \left( k_0^2 \pm \frac{\Delta k^2}{2} \right) \delta\epsilon \right]^{-1}. \quad (6)$$

The Ward identity reads

$$\langle G_c^+ \rangle - \langle G_c^- \rangle = -2 \left( \frac{\Delta k^2}{2} + i\eta \right) \langle G_c^+ G_c^- \rangle + \Delta k^2 \langle G_c^+ \delta\epsilon G_c^- \rangle. \quad (7)$$

Because of the frequency dependence of the impurity term in the classical case, these two Ward identities differ in form. They actually also indicate different conservation laws: particle conservation for electrons and energy conservation for classical waves. The Ward identity for the classical case will serve as a check of our supersymmetric formalism: In Sec. IV we derive it from the nonlinear  $\sigma$  model.

### III. NONLINEAR $\sigma$ MODEL

We derive the nonlinear  $\sigma$  model for the simplest non-trivial case: the ensemble average of a product of an advanced and a retarded Green's function. We use the notations and definitions of Ref. [5]. The advanced and retarded Green's functions can be written as integrals over supervectors

$$G^\pm(\mathbf{y}_1, \mathbf{y}_2, k^2) = \mp \frac{i}{2} \int \mathcal{D}[\Psi] \Psi_\nu(\mathbf{y}_1) \Psi_\nu^\dagger(\mathbf{y}_2) \exp[\mathcal{L}(\Psi)], \quad (8)$$

where we have omitted the index  $c$  for the Green's function of classical waves and the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} i \int d^d \mathbf{y} (\Psi^\dagger(\mathbf{y}) \{ \pm [k^2 + \Delta - \delta\epsilon(\mathbf{y})k^2] + i\eta \} \Psi(\mathbf{y})). \quad (9)$$

The quantities  $\Psi(\mathbf{x})$  are supervectors defined by

$$\Psi(\mathbf{x})^\dagger = (S^1(\mathbf{x}), S^2(\mathbf{x}), -\chi(\mathbf{x}), \chi^*(\mathbf{x})). \quad (10)$$

The quantities  $S$  are ordinary real integration variables and the  $\chi$ 's anticommute. We introduce a source term  $J(\mathbf{y}) = \text{diag}[j(\mathbf{y}), 0, 0, 0]$  in graded space and introduce the generating functional

$$Z^\pm(k^2, J) = \int \mathcal{D}[\Psi] \exp \left[ \mathcal{L} + \frac{1}{2} i \int d^d \mathbf{y} [\Psi^\dagger(\mathbf{y}) J(\mathbf{y}) \Psi(\mathbf{y})] \right]. \quad (11)$$

This functional generates the Green's function at the point  $\mathbf{y}_1 = \mathbf{y}_2$ , which is sufficient, because in the present section we are interested only in the effective action. In Sec. IV we show how to generate the Green's function with different space point arguments. The Green's function is given as functional derivative of the generating functional with respect to  $J$  at  $J=0$ ,

$$G^\pm(\mathbf{y}, \mathbf{y}) = \mp \frac{\partial Z^\pm}{\partial j(\mathbf{y})}. \quad (12)$$

We use this expression to calculate the average of the product of a retarded and an advanced Green's function taken at different frequencies  $\langle G^+ G^- \rangle$  (the two-point function). This quantity plays an important role in describing average properties of random systems, such as the level-level correlation function and the distribution function of the transmission. It serves as an example. Except for the dimension of the  $Q$  matrices appearing below and the dependence on additional frequency variables, the average  $2k$ -point function for any positive integer  $k$  is governed by an effective Lagrangian of the same type.

The generating functional  $Z$  for the two-point function is given by

$$\begin{aligned} Z(k^2, \Delta k^2, J) = & \int \mathcal{D}[\Psi] \exp \left( \frac{1}{2} i \int d^d \mathbf{y} \left[ \Psi^\dagger(\mathbf{y}) L^{1/2} \right. \right. \\ & \times \left\{ k_0^2 + \frac{\Delta k^2}{2} L + \Delta - \delta\epsilon(\mathbf{y}) \left( k_0^2 + \frac{\Delta k^2}{2} L \right) \right. \\ & \left. \left. \left. + i\eta L + J(\mathbf{y}) \right] L^{1/2} \Psi(\mathbf{y}) \right] \right), \end{aligned} \quad (13)$$

where  $L = \text{diag}(1, 1, 1, 1, -1, -1, -1, -1)$ ,  $\Psi$  are supervectors with eight components, and  $J$  is an  $8 \times 8$  matrix. All quantities are given in "advanced-retarded" notation (see Ref. [5]). Averaging over the Gaussian distribution of  $\delta\epsilon$ , we obtain the Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} i \int d^d \mathbf{y} \left\{ \Psi^\dagger(\mathbf{y}) L^{1/2} \left( k_0^2 + \frac{\Delta k^2}{2} L + \Delta + i\eta L \right) L^{1/2} \Psi(\mathbf{y}) \right. \\ & \left. - \frac{\pi}{2k_0^{d-3} \ell} \left[ \Psi^\dagger(\mathbf{y}) L^{1/2} \left( 1 + \frac{\Delta k^2}{2k_0^2} L \right) L^{1/2} \Psi(\mathbf{y}) \right]^2 \right\}. \end{aligned} \quad (14)$$

Using the Hubbard-Stratonovich transformation in the usual way and integrating over the vectors  $\Psi$ , we obtain the following form of the generating functional:

$$\begin{aligned} \bar{Z} = & \int \mathcal{D}Q \exp \left( \int d^d \mathbf{y} \left\{ -\frac{\pi\nu}{8\tau} \text{tr} Q^2 + \frac{1}{2} \text{tr} \ln \left[ k_0^2 + \frac{\Delta k^2}{2} L \right. \right. \right. \\ & \left. \left. \left. + \Delta + i\eta L + J(\mathbf{y}) - \frac{1}{2\tau} Q \left( 1 + \frac{\Delta k^2}{2k_0^2} L \right) \right] \right\} \right). \end{aligned} \quad (15)$$

Here  $\nu$  is the density of states per unit of  $k_0^2$  and per unit of volume and  $\tau = k_0^{d-3} \ell / 2\pi^2 \nu$  formally corresponds to Efetov's mean free time [3]. We have introduced these quanti-

ties in Eq. (15) in order to facilitate the direct comparison to Efetov's expression for electrons. The term  $(1/2\tau)QL(\Delta k^2/2k_0^2)$  is due to the frequency dependence of the "scattering potential"  $k^2\delta\epsilon$  in Eq. (2). Comparing Eq. (15) with the corresponding expression in Efetov's work [3], we identify (modulo factors of 2)  $\epsilon_0 k_0^2$  with the sum energy and  $\epsilon_0 \Delta k^2$  with the energy difference and find that the two expressions differ by the term  $(1/2\tau)QL(\Delta k^2/2k_0^2)$ .

To evaluate Eq. (15), we use the saddle-point approximation. This is justified if  $\tau \ll \rho$ , the mean level density. Varying the Lagrangian in Eq. (15) with respect to  $Q$  and neglecting terms proportional to  $\Delta k^2$  and source terms, we obtain the standard saddle-point equation

$$Q = \frac{1}{\pi\rho} \text{tr} \left[ k_0^2 + \Delta - \frac{1}{2\tau} Q \right]^{-1}. \quad (16)$$

This is the same equation as in the case of electrons. As in that case, the condition  $k_0 \ll 1$  (weak disorder) yields  $Q = iL$  as a solution of the saddle-point equation. The weak disorder condition also implies, however, that the term  $(1/2\tau)QL(\Delta k^2/2k_0^2)$  in Eq. (15) can be neglected. This is the case for sufficiently large  $k_0$ . Then, *there is no difference between the nonlinear  $\sigma$  models for Schrödinger waves and for classical waves*. This statement is the central result of our work. It obviously extends to the generating functionals of all higher correlation functions and thus applies universally.

The actual differences between the two theories are due to the different forms of the source terms. In the next section we show this in the case of the Ward identities.

#### IV. WARD IDENTITY

In Appendix G of Ref. [5], it was shown how a Ward identity can be derived in the context of SUSIG. We use that method to check the Ward identity (7) for classical waves, using essentially the generating functional derived in the preceding section. With slight modifications, our calculation also applies to the case of electrons. We first show how the new source terms emerge, when we introduce a new generating functional for the right-hand side (rhs) of Eq. (7). We use the coordinate representation

$$\begin{aligned} \langle \mathbf{r} | G^+ (1 - \delta\epsilon) G^- | \mathbf{r}' \rangle \\ = \int d^d \mathbf{x} G^+(\mathbf{x}, \mathbf{r}) [1 - \delta\epsilon(\mathbf{x})] G^-(\mathbf{r}', \mathbf{x}). \end{aligned} \quad (17)$$

The generating functional

$$\begin{aligned} Z_1(k^2, \Delta k^2, J_1) = - \int \mathcal{D}[\Psi_1] \exp \left\{ \frac{i}{2} \int d^d \mathbf{y} \Psi_1^\dagger(\mathbf{y}) \left[ k_0^2 + \frac{\Delta k^2}{2} \right. \right. \\ \left. \left. + \Delta - \delta\epsilon(\mathbf{y}) \left( k_0^2 + \frac{\Delta k^2}{2} \right) + i\eta \right] \Psi_1(\mathbf{y}) \right. \\ \left. \left. + \frac{i}{2} \int \int d^d \mathbf{y} d^d \mathbf{y}' \Psi_1^\dagger(\mathbf{y}) J_1(\mathbf{y}, \mathbf{y}') \Psi_1(\mathbf{y}') \right\} \end{aligned} \quad (18)$$

produces the retarded Green's function on the rhs of Eq. (17):

$$G^+(\mathbf{x}, \mathbf{r}) = \frac{\partial Z_1}{\partial j_1(\mathbf{r}, \mathbf{x})}, \quad (19)$$

where the source term  $J_1 = \text{diag}(j_1, 0, 0, 0)$  is set equal to zero after taking the derivative, while  $\Psi_1$  is a supervector with four components. We also introduce another generating functional

$$\begin{aligned} Z_2(k^2, \Delta k^2, \tilde{J}) \\ = - \int \mathcal{D}[\Psi_2] \exp \left\{ - \frac{i}{2} \int d^d \mathbf{y} \Psi_2^\dagger(\mathbf{y}) \left[ k_0^2 - \frac{\Delta k^2}{2} \right. \right. \\ \left. \left. + \Delta - \delta\epsilon(\mathbf{y}) \left( k_0^2 - \frac{\Delta k^2}{2} \right) - i\eta \right] \Psi_2(\mathbf{y}) \right. \\ \left. - \frac{i}{2} \int \int d^d \mathbf{y} d^d \mathbf{y}' \Psi_2^\dagger(\mathbf{y}) \right. \\ \left. \times [1 - \delta\epsilon(\mathbf{y})] \tilde{J}(\mathbf{y}, \mathbf{y}') \Psi_2(\mathbf{y}') \right\}. \end{aligned} \quad (20)$$

Then, immediately

$$[1 - \delta\epsilon(\mathbf{x})] G^-(\mathbf{r}', \mathbf{x}) = \frac{\partial Z_2}{\partial j_2(\mathbf{x}, \mathbf{r}')}. \quad (21)$$

Taking the product of Eqs. (18) and (20), we obtain a generating functional

$$Z_f = \int \mathcal{D}[\Psi] \exp \left[ \frac{i}{2} \mathcal{L}_f \right], \quad (22)$$

where the action is given by

$$\begin{aligned} \mathcal{L}_f = \int \int d^d \mathbf{y} d^d \mathbf{y}' \Psi^\dagger(\mathbf{y}) L^{1/2} \left\{ \left[ k_0^2 + \frac{\Delta k^2}{2} L + \Delta \right. \right. \\ \left. \left. - \delta\epsilon(\mathbf{y}) \left( k_0^2 + \frac{\Delta k^2}{2} L \right) + i\eta L \right] \delta(\mathbf{y} - \mathbf{y}') + J(\mathbf{y}, \mathbf{y}') \right. \\ \left. \left. - \delta\epsilon(\mathbf{y}') \tilde{J}(\mathbf{y}, \mathbf{y}') \right\} L^{1/2} \Psi(\mathbf{y}'), \end{aligned} \quad (23)$$

with  $\tilde{J} = \text{diag}(0, 0, 0, 0, j_2, 0, 0, 0)$  and  $J = \text{diag}(j_1, 0, 0, 0, j_2, 0, 0, 0)$ . The second partial derivatives of  $Z_f$  produce the integrands on the rhs of Eq. (7). The additional source term  $\tilde{J}(\mathbf{y}, \mathbf{y}')$  represents the important difference to the electron case. Averaging of the term containing the random part of the dielectric constant  $\delta\epsilon(\mathbf{y})$  leads to

$$\begin{aligned} \exp \left[ - \frac{1}{16\pi\nu\tau k_0^4} \int \int \int d^d \mathbf{y} d^d \mathbf{y}' d^d \mathbf{y}_1 \Psi^\dagger(\mathbf{y}) \right. \\ \left. \times L^{1/2} A(\mathbf{y}, \mathbf{y}') L^{1/2} \Psi(\mathbf{y}') \Psi^\dagger(\mathbf{y}) L^{1/2} A(\mathbf{y}, \mathbf{y}_1') L^{1/2} \Psi(\mathbf{y}_1') \right], \end{aligned} \quad (24)$$

where

$$A(\mathbf{y}, \mathbf{y}') = \left[ k_0^2 + \frac{\Delta k^2}{2} L \right] \delta(\mathbf{y} - \mathbf{y}') + \tilde{J}(\mathbf{y}, \mathbf{y}'). \quad (25)$$

To perform the Hubbard-Stratonovich transformation we introduce a supervector

$$F(\mathbf{y}) = \int d^d \mathbf{y}' A(\mathbf{y}, \mathbf{y}') L^{1/2} \Psi(\mathbf{y}'), \quad (26)$$

after which we can rewrite the expression in Eq. (24) as

$$\exp \left[ -\frac{1}{16\pi\nu\tau k_0^4} \int d^d \mathbf{y} [\Psi^\dagger(\mathbf{y}) L^{1/2} F(\mathbf{y})]^2 \right]. \quad (27)$$

Using the Hubbard-Stratonovich transformation and keeping only diffusive modes, we obtain the average of Eq. (22)

$$\begin{aligned} \bar{Z}_f &= \int \mathcal{D}[\Psi] \exp \left[ -\int d^d \mathbf{y} \frac{\pi\nu}{8\tau} \text{trg} Q^2 - \frac{1}{2} \ln \text{Detg} B(Q) \right] \\ &\equiv \int \mathcal{D}[Q] \exp[-\mathcal{L}_i(Q)], \end{aligned} \quad (28)$$

where Detg means determinant over real and graded spaces and we define a matrix

$$\begin{aligned} B(Q) &= \left( \left[ k_0^2 + \frac{\Delta k^2}{2} L + \Delta + i\eta L \right] \delta(\mathbf{y} - \mathbf{y}') + J(\mathbf{y}, \mathbf{y}') \right. \\ &\quad \left. - \frac{1}{2\tau k_0^2} Q(\mathbf{y}) \mathbf{A}(\mathbf{y}, \mathbf{y}') \right). \end{aligned} \quad (29)$$

For maximum compactness we allow  $J$  and  $\tilde{J}$  to be general symmetric  $8 \times 8$  matrices. This is permissible because we never use the particular form of the source term in the derivation in Sec. III. The saddle-point equation for this action is the same as before. Following the formalism developed in Ref. [5], we apply the transformation  $Q \rightarrow (1 + \delta T)^{-1} Q (1 + \delta T)$  (we preserve the notation of Ref. [5]), changing the action in Eq. (28) into

$$\begin{aligned} \mathcal{L}_i + \frac{1}{2} \text{Trg} B^{-1}(Q) &\left\{ \left[ \delta T, \left( \frac{\Delta k^2}{2} L + i\eta L \right) \delta(\mathbf{y} - \mathbf{y}') + J(\mathbf{y}, \mathbf{y}') \right] \right. \\ &\quad \left. - \frac{1}{2\tau} Q(\mathbf{y}) \left( \delta T, \frac{\Delta k^2}{2k_0^2} L \delta(\mathbf{y} - \mathbf{y}') + \frac{\tilde{J}(\mathbf{y}, \mathbf{y}')}{k_0^2} \right) \right\}, \end{aligned} \quad (30)$$

where Trg is the trace in both real and graded spaces. A transformation of integration variables leaves  $\bar{Z}_f$  invariant.

Therefore, terms linear in  $\delta T$  in the expression of  $\bar{Z}_f$  must vanish, which leads to the equation

$$\begin{aligned} &\int \mathcal{D}[Q] \exp(-\mathcal{L}_i) \text{Trg} B^{-1}(Q) [\delta T, J] \\ &\quad + \int \mathcal{D}[Q] \exp(-\mathcal{L}_i) \text{Trg} B^{-1}(Q) \\ &\quad \times \left[ \delta T, \left( \frac{\Delta k^2}{2} + i\eta \right) L \delta(\mathbf{y} - \mathbf{y}') \right] \\ &\quad - \frac{1}{2\tau k_0^2} \int \mathcal{D}[Q] \exp(-\mathcal{L}_i) \text{Trg} B^{-1}(Q) Q \\ &\quad \times \left\{ \delta T, \left[ \left( \frac{\Delta k^2}{2} L \delta(\mathbf{y} - \mathbf{y}') + \tilde{J}(\mathbf{y}, \mathbf{y}') \right) \right] \right\} = 0. \end{aligned} \quad (31)$$

We will consider each of these terms in detail. The first term is

$$\begin{aligned} &\int \mathcal{D}[Q] \exp(-\mathcal{L}_i) \text{Trg} [B^{-1}(\mathbf{y}, \mathbf{y}')]^{\alpha\beta} [\delta T, J]_{\mathbf{y}, \mathbf{y}'}^{\alpha\beta} \\ &= \sum_{\alpha\beta} \int \int d^d \mathbf{y} d^d \mathbf{y}' \int \mathcal{D}[Q] \exp(-\mathcal{L}_i) B_{\alpha\beta}^{-1}(Q) \\ &\quad \times [\delta T, J]^{\beta\alpha} \\ &= \sum_{kk'} \int \int d^d \mathbf{y} d^d \mathbf{y}' \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{kk'}^{(1,1)}(\mathbf{y}', \mathbf{y})} [\delta T(1,2) J(2,1) \\ &\quad - J(1,2) \delta T(2,1)]_{\mathbf{y}', \mathbf{y}}^{kk'} \\ &\quad + \sum_{kk'} \int \int d^d \mathbf{y}' d^d \mathbf{y} \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{kk'}^{(2,2)}(\mathbf{y}', \mathbf{y})} [\delta T(2,1) J(1,2) \\ &\quad - J(2,1) \delta T(1,2)]_{\mathbf{y}', \mathbf{y}}^{kk'} + \dots, \end{aligned} \quad (32)$$

where we use block notation as in Ref. [5]. The dots represent terms containing  $J(1,1)$  and  $J(2,2)$  that do not contribute to the final result. Using the explicit expressions  $\delta T_{kk'}^{1,2} = i \delta_{kk_0} \delta_{k'k'_0}$ ,  $\delta T_{kk'}^{2,1} = -i \delta_{kk_0} \delta_{k'k'_0}$ , the first term can be written as

$$\begin{aligned} &i \sum_{k'} \int \int d^d \mathbf{y} d^d \mathbf{y}' \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0 k'}^{(1,1)}(\mathbf{y}', \mathbf{y})} J_{k'_0 k'}^{(2,1)}(\mathbf{y}', \mathbf{y}) \\ &\quad + i \sum_k \int \int d^d \mathbf{y} d^d \mathbf{y}' \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{kk_0}^{(1,1)}(\mathbf{y}', \mathbf{y})} J_{kk_0}^{(1,2)}(\mathbf{y}', \mathbf{y}) - i \sum_{k'} \int \int d^d \mathbf{y} d^d \mathbf{y}' \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k'_0 k'}^{(2,2)}(\mathbf{y}', \mathbf{y})} J_{k_0 k'}^{(1,2)}(\mathbf{y}', \mathbf{y}) \\ &\quad - i \sum_k \int \int d^d \mathbf{y} d^d \mathbf{y}' \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{kk_0}^{(2,2)}(\mathbf{y}', \mathbf{y})} J_{kk_0}^{(2,1)}(\mathbf{y}', \mathbf{y}) + \dots \end{aligned} \quad (33)$$

Now taking the derivative with respect to  $J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')$ , we finally obtain

$$2i \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{1k_0}^{(1,1)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \delta_{1k_0'} - 2i \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0'1}^{(2,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \delta_{1k_0}. \quad (34)$$

We consider now the second term in Eq. (31),

$$\begin{aligned} & \int \mathcal{D}[Q] \exp(-\mathcal{L}_i) \text{Trg} B^{-1}(Q) \left[ \delta T, \left( \frac{\Delta k^2}{2} L + i \eta L \right) \delta(\mathbf{y} - \mathbf{y}') \right] = -2 \left( \frac{\Delta k^2}{2} + i \eta \right) \sum_{k, k'} \int \int d^d \mathbf{y} d^d \mathbf{y}' \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{kk'}^{(1,2)}(\mathbf{y}', \mathbf{y})} \\ & \times \delta T_{kk'}(1,2) \delta(\mathbf{y} - \mathbf{y}') + 2 \left( \frac{\Delta k^2}{2} + i \eta \right) \sum_{k, k'} \int \int d^d \mathbf{y} d^d \mathbf{y}' \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{kk'}^{(2,1)}(\mathbf{y}', \mathbf{y})} \delta T_{kk'}(2,1) \delta(\mathbf{y} - \mathbf{y}') \\ & = -2i \left( \frac{\Delta k^2}{2} + i \eta \right) \int d^d \mathbf{y} \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0 k_0'}^{(1,2)}(\mathbf{y}, \mathbf{y})} - 2i \left( \frac{\Delta k^2}{2} + i \eta \right) \int d^d \mathbf{y} \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0' k_0}^{(2,1)}(\mathbf{y}, \mathbf{y})}. \end{aligned} \quad (35)$$

The derivative with respect to  $J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')$  leads to

$$\begin{aligned} & -i \Delta k^2 \int d^d \mathbf{y} \frac{\partial^2 \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial \bar{J}_{k_0 k_0'}^{(1,2)}(\mathbf{y}, \mathbf{y}) J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \\ & - 2i \left( \frac{\Delta k^2}{2} + i \eta \right) \left( \int d^d \mathbf{y} \frac{\partial^2 \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0 k_0'}^{(1,2)}(\mathbf{y}, \mathbf{y}) \partial J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \right. \\ & \left. + \int d^d \mathbf{y} \frac{\partial^2 \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0' k_0}^{(2,1)}(\mathbf{y}, \mathbf{y}) \partial J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \right). \end{aligned} \quad (36)$$

In conclusion, the requirement that the term linear in  $\delta T$  in the expansion of  $\bar{Z}_f$  vanishes entails the equation

We expand the remaining term in Eq. (31),

$$\begin{aligned} & - \frac{1}{2\tau k_0^2} \int \mathcal{D}[Q] \exp(-\mathcal{L}_i) \text{Trg} B^{-1}(Q) Q \\ & \times \left[ \delta T, \left( \frac{\Delta k^2}{2} L \delta(\mathbf{y} - \mathbf{y}') + \bar{J}(\mathbf{y}, \mathbf{y}') \right) \right] \\ & = \sum_{\alpha, \beta} \int d^d \mathbf{y} \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial \bar{J}^{\alpha\beta}(\mathbf{y}, \mathbf{y})} \left[ \delta T, \frac{\Delta k^2}{2} L \right]_{\mathbf{y}, \mathbf{y}'}^{\alpha\beta} \\ & = -i \Delta k^2 \int d^d \mathbf{y} \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial \bar{J}_{k_0 k_0'}^{(1,2)}(\mathbf{y}, \mathbf{y})} \\ & \quad - i \Delta k^2 \int d^d \mathbf{y} \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial \bar{J}_{k_0' k_0}^{(2,1)}(\mathbf{y}, \mathbf{y})}. \end{aligned} \quad (37)$$

$$\begin{aligned} & 2 \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{1k_0}^{(1,1)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \delta_{1k_0'} - 2 \frac{\partial \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0'1}^{(2,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \delta_{1k_0} \\ & = 2 \left( \frac{\Delta k^2}{2} + i \eta \right) \int d^d \mathbf{y} \frac{\partial^2 \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0' k_0}^{(2,1)}(\mathbf{y}, \mathbf{y}) \partial J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \\ & \quad + 2i \left( \frac{\Delta k^2}{2} + i \eta \right) \int d^d \mathbf{y} \frac{\partial^2 \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial J_{k_0 k_0'}^{(1,2)}(\mathbf{y}, \mathbf{y}) \partial J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \\ & \quad + \Delta k^2 \int d^d \mathbf{y} \frac{\partial^2 \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial \bar{J}_{k_0 k_0'}^{(1,2)}(\mathbf{y}, \mathbf{y}) J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0} \\ & \quad + \Delta k^2 \int d^d \mathbf{y} \frac{\partial^2 \bar{Z}_f(k_0^2, \Delta k^2, J)}{\partial \bar{J}_{k_0' k_0}^{(2,1)}(\mathbf{y}, \mathbf{y}) J_{11}^{(1,2)}(\mathbf{x}, \mathbf{x}')} \Big|_{J, \bar{J}=0}. \end{aligned} \quad (38)$$

The term proportional to  $\bar{J}$  is omitted because it does not contribute to the final result. Taking the derivative with respect to  $J_{1,1}^{(1,2)}$ , we find

Replacing  $\bar{Z}_f$  everywhere by  $Z$ , using the definition of  $Z$ , and setting  $k_0 = k_0' = 1$ , we immediately obtain a Ward identity in the form of Eq. (7). Indeed, Eq. (34) corresponds to the left-hand side of Eq. (7), whereas Eqs. (36) and (38) lead to the rhs of Eq. (7).

## V. CONCLUSIONS

We have derived a nonlinear supersymmetric  $\sigma$  model for classical scalar waves. We have shown that in the weak disorder limit ( $k_0 \ell \gg 1$ ), the effective Lagrangian of this model is identical to the one for electrons. In this limit, the main difference between the wave equations for classical and Schrödinger waves, the frequency dependence of the random potential, does not lead to different wave behavior. We have also shown that the Ward identities for classical and for Schrödinger waves are both fulfilled by the same effective Lagrangian. This is due to the different source terms. Outside the regime of weak disorder, i.e., at low frequencies, the

frequency dependence of the disorder potential for classical waves suppresses disorder effects altogether. This does not happen for electrons. This low-frequency domain is not accessible to the nonlinear  $\sigma$  model.

## ACKNOWLEDGMENTS

V.K. appreciates very useful discussions with Dr. A. Müller-Groeling and Dr. Y. Fyodorov. V.K. gratefully acknowledges the financial support of MINERVA. B.E. wishes to thank Dr. A. Müller-Groeling and Professor A. Nourreddine for valuable discussions.

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